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A discrete-time relativistic Toda lattice

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Abstract. For each of the two simplest Hamiltonian flows from the relativistic Toda hierarchy we introduce two integrable symplectic discretizations. All four discrete-time systems are demonstrated to belong to the same hierarchy and to exemplify the general scheme for symplectic maps on groups equipped with quadratic Poisson brackets. The initial-value problem for the difference equations is solved in terms of a factorization problem in a group. Interpolating Hamiltonian flows are found for all maps.

1. Introduction

Although the subject of integrable symplectic maps has received considerable attention in recent years, order in this area still seems to be lacking. Given an integrable system of ordinary differential equations with such attributes as a Lax pair and an r-matrix, one would like to construct its difference approximation, preferably also with (a discrete-time analogue of) a Lax pair, an r-matrix, etc. Recent years have produced several successful examples of such a construction [1–8], but still not the general rules and recipes, not to mention algorithms.

Recently there appeared for the first time examples where the Lax matrix of the discrete-time approximation *coincides* with the Lax matrix of the continuous-time system, so the discrete-time system belongs to the *same* integrable hierarchy as the underlying continuous-time one (systems of Calogero–Moser type [7, 8]). We want to present here one more example of this type, which can be studied in full (and beautiful) detail—the discrete-time analogue of the relativistic Toda lattice [9]; see also [10–12].

The paper is organized as follows. In section 2 and section 3 we start by recalling some facts about the continuous-time relativistic Toda lattice, its r-matrix structure and the solution in terms of a factorization problem in a matrix group. Most of these facts are by now well known, but it has turned out to be rather difficult or even impossible to find them in the literature in a form suitable for our present purposes. In section 4 we go on to introduce the equations of motion for the four versions of the discrete-time relativistic Toda lattice and derive their Lax representations. Section 5 is devoted to the Newtonian form of equations of motion. Finally, in section 6 we discuss some general aspects of integrable discretizations.

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2. The relativistic Toda lattice

We consider in this paper two flows from the relativistic Toda hierarchy. The first is described by the equations of motion

$$\dot{d}_k = d_k(c_k - c_{k-1}) \qquad \dot{c}_k = c_k(d_{k+1} + c_{k+1} - d_k - c_{k-1}) \qquad 1 \leqslant k \leqslant N \tag{2.1}$$

and the second is described by the equations of motion

$$\dot{d}_k = d_k\left(\frac{c_k}{d_k d_{k+1}} - \frac{c_{k-1}}{d_{k-1}d_k}\right) \qquad \dot{c}_k = c_k\left(\frac{1}{d_k} - \frac{1}{d_{k+1}}\right) \qquad 1 \leqslant k \leqslant N. \tag{2.2}$$

Both of these sets of equations may be considered either under open-end boundary conditions ($d_{N+1} = c_0 = c_N = 0$), or under periodic ones (all the subscripts are taken mod N, so that $d_{N+1} = d_1$, $c_0 = c_N$, $c_{N+1} = c_1$). Both flows are Hamiltonian with respect to the Poisson brackets

$$\{c_k, c_{k+1}\} = -c_k c_{k+1} \qquad \{c_k, d_{k+1}\} = -c_k d_{k+1} \qquad \{c_k, d_k\} = c_k d_k \tag{2.3}$$

(only the non-vanishing brackets are written down), with Hamiltonian functions

$$J_+ = \sum_{k=1}^{N}(d_k + c_k) \qquad J_- = \sum_{k=1}^{N}\frac{d_k + c_k}{d_k d_{k+1}}$$

respectively.

The Lax representation and the integrability for the flows (2.1) and (2.2) are dealt with in the following statement. Introduce two N-by-N matrices depending on the phase-space coordinates c_k, d_k and (in the periodic case) on the additional parameter λ:

$$L(c, d, \lambda) = \sum_{k=1}^{N} d_k E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1,k} \tag{2.4}$$

$$U(c, d, \lambda) = \sum_{k=1}^{N} E_{kk} - \lambda^{-1} \sum_{k=1}^{N} c_k E_{k,k+1}. \tag{2.5}$$

Here E_{jk} stands for the matrix whose only non-zero entry at the intersection of the jth row and the kth column is equal to 1. In the periodic case we have $E_{N+1,N} = E_{1,N}$, $E_{N,N+1} = E_{N,1}$; in the open-end case we set $\lambda = 1$, and $E_{N+1,N} = E_{N,N+1} = 0$. Consider also the following two matrices:

$$T_+(c, d, \lambda) = L(c, d, \lambda)U^{-1}(c, d, \lambda) \qquad T_-(c, d, \lambda) = U^{-1}(c, d, \lambda)L(c, d, \lambda). \tag{2.6}$$

Theorem 1. The flow (2.1) is equivalent to the following matrix differential equations:

$$\dot{L} = LB - AL \qquad \dot{U} = UB - AU$$

which imply also

$$\dot{T}_+ = [T_+, A] \qquad \dot{T}_- = [T_-, B]$$

where

$$A(c, d, \lambda) = \sum_{k=1}^{N}(d_k + c_{k-1})E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1,k} \tag{2.7}$$

$$B(c, d, \lambda) = \sum_{k=1}^{N}(d_k + c_k)E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1,k}. \tag{2.8}$$

The flow (2.2) is equivalent to the following matrix differential equations:

$$\dot{L} = LD - CL \qquad \dot{U} = UD - CU$$

which imply also

$$\dot{T}_+ = \left[T_+, C\right] \qquad \dot{T}_- = \left[T_-, D\right]$$

where

$$C(c, d, \lambda) = -\lambda^{-1} \sum_{k=1}^{N} \frac{c_k}{d_{k+1}} E_{k,k+1} \tag{2.9}$$

$$D(c, d, \lambda) = -\lambda^{-1} \sum_{k=1}^{N} \frac{c_k}{d_k} E_{k,k+1}. \tag{2.10}$$

The spectral invariants of the matrices $T_\pm(c, d, \lambda)$ serve as integrals of motion for both of the flows (2.1) and (2.2). These integrals are in involution with respect to the Poisson bracket (2.3).

So we see that either of the matrices $T_\pm$ (they are in fact connected by means of a similarity transformation) can serve as a Lax matrix for *both* of the flows (2.1) and (2.2). Note also that Hamiltonians $J_\pm$ belong to the set of invariant functions of $T_\pm$, as it is easy to check that

$$J_+ = \mathrm{tr}(T_\pm) \qquad J_- = \mathrm{tr}(T_\pm^{-1}).$$

3. Algebraic structure

Here we recall some of the results of [11, 12] on the algebraic interpretation of the relativistic Toda lattice as a Hamiltonian system on a particular orbit of a certain Poisson bracket on a matrix group ([11] deals with a gauge transformation of a Lax matrix, which results in a different Poisson bracket on a group). The results concerning the difference Lax triads (part (c) of theorem 2 below) are, as far as I am aware, new; however, similar results for less general Poisson brackets can be found in [13, 14]. (Recall that the orbit of the relativistic Toda lattice fails to be a Poisson submanifold for the brackets from [13, 14], so the generalization in [12] is necessary.) Theorem 2 serves as a wide generalization of seminal work by Symes [15].

First of all, we define the relevant algebras, groups and decompositions.

(1) For the open-end case we set $\mathbf{g} = gl(N)$. As a linear space, $\mathbf{g}$ may be represented as a direct sum of two subspaces, which serve also as subalgebras: $\mathbf{g} = \mathbf{g}_+ \oplus \mathbf{g}_-$, where $\mathbf{g}_+$ ($\mathbf{g}_-$) is a space of all lower triangular (strictly upper triangular) N-by-N matrices. The corresponding groups are: $\mathbf{G} = GL(N)$; $\mathbf{G}_+$ ($\mathbf{G}_-$) is a group of all non-degenerate lower triangular N-by-N matrices (upper triangular N-by-N matrices with unities on the diagonal).

(2) For the periodic case $\mathbf{g}$ is a certain twisted-loop algebra over $gl(N)$:

$$\mathbf{g} = \left\{\tau(\lambda) \in gl(N)[\lambda, \lambda^{-1}]\colon \Omega\tau(\lambda)\Omega^{-1} = \tau(\omega\lambda)\right\}$$

where $\Omega = \mathrm{diag}(1, \omega, \ldots, \omega^{N-1})$, $\omega = \exp(2\pi \mathrm{i}/N)$. Again, as a linear space $\mathbf{g} = \mathbf{g}_+ \oplus \mathbf{g}_-$, where $\mathbf{g}_+$ ($\mathbf{g}_-$) is a subspace and subalgebra consisting of $\tau(\lambda)$ containing only non-negative (only negative) powers of λ. The corresponding groups are: $\mathbf{G}$, the twisted-loop group, i.e. the group of $GL(N)$-valued functions $T(\lambda)$ of the complex parameter λ, regular in $\mathbb{C}P^1 \backslash \{0, \infty\}$ and satisfying $\Omega T(\lambda)\Omega^{-1} = T(\omega\lambda)$; $\mathbf{G}_+$ ($\mathbf{G}_-$) is the subgroup consisting of

$T(\lambda)$ regular in the neighbourhood of $\lambda = 0$ (regular in the neighbourhood of $\lambda = \infty$ and taking the value I in $\lambda = \infty$).

For both the open-end case and the periodic case every $\tau \in \mathbf{g}$ admits a unique decomposition $\tau = l - u$, where $l \in \mathbf{g}_+$, $u \in \mathbf{g}_-$. We use the notation $l = \pi_+(\tau)$, $u = \pi_-(\tau)$. Analogously, for both of the cases every $T \in \mathbf{G}$ from some neighbourhood of the group unity admits a unique factorization $T = \mathcal{L}\mathcal{U}^{-1}$, where $\mathcal{L} \in \mathbf{G}_+$, $\mathcal{U} \in \mathbf{G}_-$. We denote the factors as $\mathcal{L} = \Pi_+(T)$, $\mathcal{U} = \Pi_-(T)$.

Recall also that the derivative $\mathrm{d}\varphi(T) \in \mathbf{g}$ of the conjugation-invariant function φ: $\mathbf{G} \mapsto \mathbb{C}$ is defined by the relation

$$\mathrm{tr}(\mathrm{d}\varphi(T)u) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\varphi(T\mathrm{e}^{\varepsilon u})\Big|_{\varepsilon=0} = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\varphi(\mathrm{e}^{\varepsilon u}T)\Big|_{\varepsilon=0} \qquad \forall u \in \mathbf{g}.$$

Theorem 2.

(a) Equip $\mathbf{G} \times \mathbf{G}$ with the quadratic Poisson bracket (38)–(41) from [12], and $\mathbf{G}$ with the quadratic Poisson bracket (33) from [12]. Then the set of pairs of matrices $\{(L(c,d,\lambda), U(c,d,\lambda))\}$ forms a Poisson submanifold in $\mathbf{G} \times \mathbf{G}$, the set of matrices $\{T_\pm(c,d,\lambda)\}$ forms a Poisson submanifold in $\mathbf{G}$, and the maps $(L,U) \mapsto T_+ = LU^{-1}$ and $(L,U) \mapsto T_- = U^{-1}L$ are Poisson maps from $\mathbf{G} \times \mathbf{G}$ into $\mathbf{G}$.

(b) Let φ: $\mathbf{G} \mapsto \mathbb{C}$ be an invariant function on $\mathbf{G}$. Then the Hamiltonian flow on $\mathbf{G} \times \mathbf{G}$ with the Hamiltonian function $\varphi(LU^{-1}) = \varphi(U^{-1}L)$ has the form

$$\dot{L} = L\pi_\pm(\mathrm{d}\varphi(T_-)) - \pi_\pm(\mathrm{d}\varphi(T_+))L$$
$$\dot{U} = U\pi_\pm(\mathrm{d}\varphi(T_-)) - \pi_\pm(\mathrm{d}\varphi(T_+))U$$

and the Hamiltonian flow on $\mathbf{G}$ with the Hamiltonian function $\varphi(T)$ has the form

$$\dot{T} = \big[T, \pi_\pm(\mathrm{d}\varphi(T))\big] \qquad T = T_+ \text{ or } T_-.$$

These flows admit the following solution in terms of the factorization problem:

$$\mathrm{e}^{t\,\mathrm{d}\varphi(T_\pm(0))} = \mathcal{L}_\pm(t)\mathcal{U}_\pm^{-1}(t) \qquad \mathcal{L}_\pm(t) \in \mathbf{G}_+ \qquad \mathcal{U}_\pm(t) \in \mathbf{G}_-$$

(this problem has solutions at least for sufficiently small t):

$$L(t) = \mathcal{L}_+^{-1}(t)L(0)\mathcal{L}_-(t) = \mathcal{U}_+^{-1}(t)L(0)\mathcal{U}_-(t)$$
$$U(t) = \mathcal{L}_+^{-1}(t)U(0)\mathcal{L}_-(t) = \mathcal{U}_+^{-1}(t)U(0)\mathcal{U}_-(t)$$

and so

$$T_\pm(t) = \mathcal{L}_\pm^{-1}(t)T_\pm(0)\mathcal{L}_\pm(t) = \mathcal{U}_\pm^{-1}(t)T_\pm(0)\mathcal{U}_\pm(t).$$

(c) Let f: $\mathbf{G} \mapsto \mathbf{G}$ be a conjugation-covariant function (so $\log(f)$: $\mathbf{G} \mapsto \mathbf{g}$ is a derivative of a conjugation-invariant function on $\mathbf{G}$). Then the system of difference equations ($t \in h\mathbb{Z}$)

$$L(t+h) = \Pi_\pm^{-1}\Big(f(T_+(t))\Big)L(t)\Pi_\pm\Big(f(T_-(t))\Big)$$
$$U(t+h) = \Pi_\pm^{-1}\Big(f(T_+(t))\Big)U(t)\Pi_\pm\Big(f(T_-(t))\Big)$$

defines a Poisson map $\mathbf{G} \times \mathbf{G} \mapsto \mathbf{G} \times \mathbf{G}$, and the difference equation

$$T(t+h) = \Pi_\pm^{-1}\Big(f(T(t))\Big)T(t)\Pi_\pm\Big(f(T(t))\Big) \qquad T = T_+ \text{ or } T_-$$

defines a Poisson map $\mathbf{G} \mapsto \mathbf{G}$. These difference equations admit the following solution in terms of the factorization problem:

$$f^n(T_\pm(0)) = \mathcal{L}_\pm(nh)\mathcal{U}_\pm^{-1}(nh) \qquad \mathcal{L}_\pm(nh) \in \mathbf{G}_+ \qquad \mathcal{U}_\pm(nh) \in \mathbf{G}_-$$

(this problem has solutions for a given n at least if $f(T_\pm(0))$ is sufficiently close to the group unity I):

$$L(nh) = \mathcal{L}_+^{-1}(nh)L(0)\mathcal{L}_-(nh) = \mathcal{U}_+^{-1}(nh)L(0)\,\mathcal{U}_-(nh)$$
$$U(nh) = \mathcal{L}_+^{-1}(nh)U(0)\mathcal{L}_-(nh) = \mathcal{U}_+^{-1}(nh)U(0)\,\mathcal{U}_-(nh)$$

and so

$$T_\pm(nh) = \mathcal{L}_\pm^{-1}(nh)T_\pm(0)\mathcal{L}_\pm(nh) = \mathcal{U}_\pm^{-1}(nh)T_\pm(0)\,\mathcal{U}_\pm(nh).$$

(d) The solutions of the difference equations of part (c) are interpolated by the flows of part (b) with the Hamiltonian function $\varphi(T)$ defined by

$$\mathrm{d}\varphi(T) = h^{-1}\log(f(T)).$$

(In part (c) and below, $\Pi_\pm^{-1}(f)$ stands for $(\Pi_\pm(f))^{-1}$.)

Part (b) of the last theorem explains, in particular, theorem 1, as for $J_+(T) = \mathrm{tr}(T)$, $J_-(T) = \mathrm{tr}(T^{-1})$ we have

$$\mathrm{d}J_+(T) = T \qquad \mathrm{d}J_-(T) = -T^{-1}$$

and it is not hard to check that

$$A = \pi_+(T_-) \qquad B = \pi_+(T_+) \qquad C = \pi_-(-T_-^{-1}) \qquad D = \pi_-(-T_+^{-1}).$$

4. Integrable maps for the relativistic Toda lattice

We proceed now to define and investigate the four integrable maps related to the $J_\pm$-flows of the relativistic Toda hierarchy. All of them depend on the time-step $h > 0$ as a parameter.

4.1. The first map

Consider the recurrence relations

$$\mathfrak{a}_k = 1 + hd_k + \frac{hc_{k-1}}{\mathfrak{a}_{k-1}} \qquad 1 \leqslant k \leqslant N. \tag{4.1}$$

In the open-end case, because $c_0 = 0$ we obtain from (4.1) the following finite-continued-fractions expressions for $\mathfrak{a}_k = \mathfrak{a}_k(c, d)$:

$$\mathfrak{a}_1 = 1 + hd_1 \qquad \mathfrak{a}_2 = 1 + hd_2 + \frac{hc_1}{1 + hd_1} \qquad \ldots$$

$$\mathfrak{a}_N = 1 + hd_N + \cfrac{hc_{N-1}}{1 + hd_{N-1} + \cfrac{hc_{N-2}}{1 + hd_{N-2} + \ddots + \cfrac{hc_1}{1 + hd_1}}}.$$

Obviously, they satisfy the asymptotic relation

$$\mathfrak{a}_k = 1 + h(d_k + c_{k-1}) + \mathrm{O}(h^2) \qquad 1 \leqslant k \leqslant N. \tag{4.2}$$

In the periodic case the recurrences (4.1) uniquely define the $\mathfrak{a}_k$ as N-periodic infinite continued fractions. It can be proved that these continued fractions converge and their values satisfy (4.2).

Define other functions $\mathfrak{b}_k(c, d)$ by

$$\mathfrak{b}_k = \mathfrak{a}_k \frac{\mathfrak{a}_{k+1} - hd_{k+1}}{\mathfrak{a}_k - hd_k} = \mathfrak{a}_{k-1} \frac{\mathfrak{a}_k + hc_k}{\mathfrak{a}_{k-1} + hc_{k-1}} \qquad 1 \leqslant k \leqslant N. \tag{4.3}$$

Note that the compatibility of these two expressions for $\mathfrak{b}_k$ is an immediate consequence of (4.1), and that it follows from (4.2) that

$$\mathfrak{b}_k = 1 + h(d_k + c_k) + \mathrm{O}(h^2) \qquad 1 \leqslant k \leqslant N. \tag{4.4}$$

Consider the discrete-time dynamical system defined by the map

$$\widetilde{d}_k = d_k \frac{\mathfrak{a}_{k+1} - hd_{k+1}}{\mathfrak{a}_k - hd_k} \qquad \widetilde{c}_k = c_k \frac{\mathfrak{a}_{k+1} + hc_{k+1}}{\mathfrak{a}_k + hc_k} \tag{4.5}$$

(here we adopt the notation from [7] and [8], according to which the tilde denotes the time shift—so, for instance, $\widetilde{d}_k$ stands for $d_k(t + h)$, if $d_k = d_k(t)$).

From (4.2) it is evident that this map serves as a finite-difference approximation to the flow (2.1). This map turns out to be integrable and to admit a Lax representation, described in the next theorem and involving two matrices

$$\mathbf{A}(c, d, \lambda) = \sum_{k=1}^{N} \mathfrak{a}_k E_{kk} + h\lambda \sum_{k=1}^{N} E_{k+1,k} \tag{4.6}$$

$$\mathbf{B}(c, d, \lambda) = \sum_{k=1}^{N} \mathfrak{b}_k E_{kk} + h\lambda \sum_{k=1}^{N} E_{k+1,k} \tag{4.7}$$

(we adopt here the conventions formulated after the formulae (2.4) and (2.5)).

Theorem 3.

(i) $\mathbf{A}(c, d, \lambda) = \Pi_+ \left(I + hT_+(c, d, \lambda)\right)$.

(ii) $\mathbf{B}(c, d, \lambda) = \Pi_+ \left(I + hT_-(c, d, \lambda)\right)$.

(iii) The dynamical system (4.5) admits a Lax representation

$$\mathbf{A}(t)L(t + h) = L(t)\mathbf{B}(t) \qquad \mathbf{A}(t)U(t + h) = U(t)\mathbf{B}(t) \tag{4.8}$$

which implies also

$$T_+(t + h) = \mathbf{A}^{-1}(t)T_+(t)\mathbf{A}(t) \qquad T_-(t + h) = \mathbf{B}^{-1}(t)T_-(t)\mathbf{B}(t). \tag{4.9}$$

Proof 1. Note first of all that the recurrence relations (4.1) are equivalent to the matrix decomposition

$$U(c, d, \lambda) + hL(c, d, \lambda) = \mathbf{A}(c, d, \lambda)P_1(c, d, \lambda) \tag{4.10}$$

where

$$P_1(c, d, \lambda) = \sum_{k=1}^{N} E_{kk} - \lambda^{-1} \sum_{k=1}^{N} \frac{c_k}{\mathfrak{a}_k} E_{k,k+1} \in \mathbf{G}_-.$$

Now the first statement of the theorem follows immediately, because

$$I + hT_+ = \mathbf{A}P_1U^{-1} \qquad \text{and} \qquad P_1U^{-1} \in \mathbf{G}_-.$$

Turning to the third statement, note that the two matrix equations in (4.8) are equivalent to the following two pairs of equations, respectively:

$$\mathfrak{a}_k\widetilde{d}_k = d_k\mathfrak{b}_k \qquad h\widetilde{d}_k + \mathfrak{a}_{k+1} = hd_{k+1} + \mathfrak{b}_k \tag{4.11}$$

$$\mathfrak{a}_k\widetilde{c}_k = c_k\mathfrak{b}_{k+1} \qquad h\widetilde{c}_k - \mathfrak{a}_{k+1} = hc_{k+1} - \mathfrak{b}_{k+1}. \tag{4.12}$$

Now (4.11) can immediately be shown to be equivalent to the first equation in (4.5) together with the first expression for $\mathfrak{b}_k$ in (4.3), and (4.12) is equivalent to the second equation in (4.5) together with the second expression for $\mathfrak{b}_k$ in (4.3).

Now we can prove the second statement of the theorem. Indeed, equations (4.10) and (4.8) imply:

$$I + hT_- = U^{-1}\mathbf{A}P_1 = \mathbf{B}\widetilde{U}^{-1}P_1 \qquad \text{and} \qquad \widetilde{U}^{-1}P_1 \in \mathbf{G}_-.$$

The theorem is proved. □

4.2. The second map

This time the relevant recurrence relations read

$$\frac{c_k}{\mathfrak{d}_k} = d_k - h - h\mathfrak{d}_{k-1} \qquad 1 \leqslant k \leqslant N. \tag{4.13}$$

As before, in the open-end case because $\mathfrak{d}_0 = 0$ we obtain the closed expressions

$$\mathfrak{d}_1 = \frac{c_1}{d_1 - h} \qquad \mathfrak{d}_2 = \frac{c_2}{d_2 - h - \dfrac{hc_1}{d_1 - h}} \qquad \ldots$$

$$\mathfrak{d}_{N-1} = \cfrac{c_{N-1}}{d_{N-1} - h - \cfrac{hc_{N-2}}{d_{N-2} - h - \ddots + \cfrac{hc_1}{d_1 - h}}}.$$

Obviously, we have

$$\mathfrak{d}_k = \frac{c_k}{d_k} + \mathrm{O}(h) \qquad 1 \leqslant k \leqslant N. \tag{4.14}$$

In the periodic case the recurrence relations (4.13) uniquely define the $\mathfrak{d}_k$ as the N-periodic infinite continued fractions, which again converge and whose values satisfy relation (4.14).

Define also the functions

$$\mathfrak{c}_k = \mathfrak{d}_k \frac{d_k - h\mathfrak{d}_{k-1}}{d_{k+1} - h\mathfrak{d}_k} = \mathfrak{d}_{k+1} \frac{c_k + h\mathfrak{d}_k}{c_{k+1} + h\mathfrak{d}_{k+1}} \qquad 1 \leqslant k \leqslant N. \tag{4.15}$$

Again, the compatibility of two expressions for $\mathfrak{c}_k$ is a direct consequence of (4.13). It follows from (4.14) that

$$\mathfrak{c}_k = \frac{c_k}{d_{k+1}} + \mathrm{O}(h) \qquad 1 \leqslant k \leqslant N. \tag{4.16}$$

Now consider the discrete-time dynamical system defined by the map

$$\widetilde{d}_k = d_{k+1} \frac{d_k - h\mathfrak{d}_{k-1}}{d_{k+1} - h\mathfrak{d}_k} \qquad \widetilde{c}_k = c_{k+1} \frac{c_k + h\mathfrak{d}_k}{c_{k+1} + h\mathfrak{d}_{k+1}}. \tag{4.17}$$

In view of (4.14) it is obvious that this map is a difference approximation to the flow (2.2).

The Lax representation for this map is described in terms of the following matrices:

$$\mathbf{C}(c, d, \lambda) = \sum_{k=1}^{N} E_{kk} + h\lambda^{-1} \sum_{k=1}^{N} \mathfrak{c}_k E_{k,k+1} \tag{4.18}$$

$$\mathbf{D}(c, d, \lambda) = \sum_{k=1}^{N} E_{kk} + h\lambda^{-1} \sum_{k=1}^{N} \mathfrak{d}_k E_{k,k+1}. \tag{4.19}$$

Theorem 4.

(i) $\mathbf{D}(c,d,\lambda) = \Pi_-^{-1}\big(I - hT_-^{-1}(c,d,\lambda)\big)$.

(ii) $\mathbf{C}(c,d,\lambda) = \Pi_-^{-1}\big(I - hT_+^{-1}(c,d,\lambda)\big)$.

(iii) The dynamical system (4.17) admits a Lax representation

$$L(t+h)\mathbf{D}(t) = \mathbf{C}(t)L(t) \qquad U(t+h)\mathbf{D}(t) = \mathbf{C}(t)U(t) \tag{4.20}$$

which implies also

$$T_+(t+h) = \mathbf{C}(t)T_+(t)\mathbf{C}^{-1}(t) \qquad T_-(t+h) = \mathbf{D}(t)T_-(t)\mathbf{D}^{-1}(t). \tag{4.21}$$

Proof 2. The scheme of the proof is the same as for theorem 3, so we give here only the necessary formulae. The recurrence relations (4.13) are equivalent to the matrix decomposition

$$L(c,d,\lambda) - hU(c,d,\lambda) = P_2(c,d,\lambda)\mathbf{D}(c,d,\lambda) \tag{4.22}$$

with

$$P_2(c,d,\lambda) = \sum_{k=1}^{N} \frac{c_k}{\mathfrak{d}_k} E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1,k} \in \mathbf{G}_+.$$

This implies the first, and, together with (4.20), the second statement of the theorem, because

$$I - hT_-^{-1} = L^{-1}P_2\mathbf{D} \qquad \text{and} \qquad L^{-1}P_2 \in \mathbf{G}_+$$

$$I - hT_+^{-1} = P_2\mathbf{D}L^{-1} = P_2\widetilde{L}^{-1}\mathbf{C} \qquad \text{and} \qquad P_2\widetilde{L}^{-1} \in \mathbf{G}_+.$$

The third statement follows immediately from the representation of (4.20) as

$$\widetilde{d}_k\mathfrak{d}_k = \mathfrak{c}_k d_{k+1} \qquad \widetilde{d}_k + h\mathfrak{d}_{k-1} = d_k + h\mathfrak{c}_k \tag{4.23}$$

$$\widetilde{c}_k\mathfrak{d}_{k+1} = \mathfrak{c}_k c_{k+1} \qquad \widetilde{c}_k - h\mathfrak{d}_k = c_k - h\mathfrak{c}_k \tag{4.24}$$

which is equivalent to (4.17) and (4.15). This proves the theorem. □

4.3. *The third map*

This time we introduce the two relevant sequences via the formulae

$$\beta_k = 1 - hd_k - \frac{hc_k}{\beta_{k+1}} \qquad 1 \leqslant k \leqslant N \tag{4.25}$$

$$\alpha_k = \beta_k \frac{\beta_{k-1} + hd_{k-1}}{\beta_k + hd_k} = \beta_{k+1}\frac{\beta_k - hc_{k-1}}{\beta_{k+1} - hc_k} \qquad 1 \leqslant k \leqslant N. \tag{4.26}$$

As before, in the open-end case the equality $c_N = 0$ leads to closed expressions for the β_k:

$$\beta_N = 1 - hd_N \qquad \beta_{N-1} = 1 - hd_{N-1} - \frac{hc_{N-1}}{1 - hd_N} \qquad \ldots$$

$$\beta_1 = 1 - hd_1 - \cfrac{hc_1}{1 - hd_2 - \cfrac{hc_2}{1 - hd_3 - \ddots + \cfrac{hc_{N-1}}{1 - hd_N}}}$$

satisfying

$$\beta_k = 1 - h(d_k + c_k) + \mathrm{O}(h^2) \qquad 1 \leqslant k \leqslant N \tag{4.27}$$

which together with (4.26) implies also

$$\alpha_k = 1 - h(d_k + c_{k-1}) + \mathrm{O}(h^2) \qquad 1 \leqslant k \leqslant N. \tag{4.28}$$

In the periodic case the recurrence relations (4.25) uniquely define the β_k as the N-periodic infinite continued fractions, which again can be proved to converge and to satisfy (4.27).

Now we introduce the map generating a new discrete-time dynamical system:

$$\widetilde{d}_k = d_k \frac{\beta_{k-1} + hd_{k-1}}{\beta_k + hd_k} \qquad \widetilde{c}_k = c_k \frac{\beta_k - hc_{k-1}}{\beta_{k+1} - hc_k}. \tag{4.29}$$

Because of (4.27) this map approximates the flow (2.1). The two matrices participating in its Lax representation are

$$\mathcal{A}(c, d, \lambda) = \sum_{k=1}^{N} \alpha_k E_{kk} - h\lambda \sum_{k=1}^{N} E_{k+1,k} \tag{4.30}$$

$$\mathcal{B}(c, d, \lambda) = \sum_{k=1}^{N} \beta_k E_{kk} - h\lambda \sum_{k=1}^{N} E_{k+1,k}. \tag{4.31}$$

Theorem 5.

(i) $\mathcal{B}(c, d, \lambda) = \Pi_+^{-1}\Big((I - hT_-(c, d, \lambda))^{-1}\Big)$.

(ii) $\mathcal{A}(c, d, \lambda) = \Pi_+^{-1}\Big((I - hT_+(c, d, \lambda))^{-1}\Big)$.

(iii) The dynamical system (4.29) admits a Lax representation

$$L(t + h)\mathcal{B}(t) = \mathcal{A}(t)L(t) \qquad U(t + h)\mathcal{B}(t) = \mathcal{A}(t)U(t) \tag{4.32}$$

which implies also

$$T_+(t + h) = \mathcal{A}(t)T_+(t)\mathcal{A}^{-1}(t) \qquad T_-(t + h) = \mathcal{B}(t)T_-(t)\mathcal{B}^{-1}(t). \tag{4.33}$$

Proof 3. The recurrence relations (4.25) are this time equivalent to the matrix factorization

$$U(c, d, \lambda) - hL(c, d, \lambda) = P_3(c, d, \lambda)\mathcal{B}(c, d, \lambda) \tag{4.34}$$

where

$$P_3(c, d, \lambda) = \sum_{k=1}^{N} E_{kk} - \lambda^{-1} \sum_{k=1}^{N} \frac{c_k}{\beta_{k+1}} E_{k,k+1} \in \mathbf{G}_-.$$

This implies (i) immediately and, together with (4.32), also implies (ii), because

$$I - hT_- = U^{-1}P_3\mathcal{B} \qquad \text{and} \qquad U^{-1}P_3 \in \mathbf{G}_-$$

$$I - hT_+ = P_3\mathcal{B}U^{-1} = P_3\widetilde{U}^{-1}\mathcal{A} \qquad \text{and} \qquad P_3\widetilde{U}^{-1} \in \mathbf{G}_-.$$

In order to prove (iii) we represent (4.32) as

$$\widetilde{d}_k\beta_k = \alpha_k d_k \qquad h\widetilde{d}_k - \beta_{k-1} = hd_{k-1} - \alpha_k \tag{4.35}$$

$$\widetilde{c}_k\beta_{k+1} = \alpha_k c_k \qquad h\widetilde{c}_k + \beta_k = hc_{k-1} + \alpha_k \tag{4.36}$$

which is readily checked to be equivalent to (4.29) and (4.26). The theorem is proved. □

4.4. The fourth map

The last pair of sets of auxiliary functions is defined by

$$\frac{c_{k-1}}{\gamma_{k-1}} = d_k + h + h\gamma_k \qquad 1 \leqslant k \leqslant N \tag{4.37}$$

$$\delta_k = \gamma_k \frac{d_{k+1} + h\gamma_{k+1}}{d_k + h\gamma_k} = \gamma_{k-1} \frac{c_k - h\gamma_k}{c_{k-1} - h\gamma_{k-1}} \qquad 1 \leqslant k \leqslant N. \tag{4.38}$$

In the open-end case, because $c_N = 0$ we obtain as usual the closed relations

$$\gamma_{N-1} = \frac{c_{N-1}}{d_N + h} \qquad \gamma_{N-2} = \frac{c_{N-2}}{d_{N-1} + h + \dfrac{hc_{N-1}}{d_N + h}} \qquad \ldots$$

$$\gamma_1 = \cfrac{c_1}{d_2 + h + \cfrac{hc_2}{d_3 + h + \ddots + \cfrac{hc_{N-1}}{d_N + h}}}.$$

Obviously, we have

$$\gamma_k = \frac{c_k}{d_{k+1}} + \mathrm{O}(h) \qquad 1 \leqslant k \leqslant N \tag{4.39}$$

which implies also

$$\delta_k = \frac{c_k}{d_k} + \mathrm{O}(h) \qquad 1 \leqslant k \leqslant N. \tag{4.40}$$

In the periodic case the recurrence relations (4.37) uniquely define the γ_k as the N-periodic infinite continued fractions, whose convergence can be proved, as can the validity of (4.39).

The last discrete-time dynamical system that we introduce in this paper is defined by the map

$$\widetilde{d}_k = d_{k-1} \frac{d_k + h\gamma_k}{d_{k-1} + h\gamma_{k-1}} \qquad \widetilde{c}_k = c_{k-1} \frac{c_k - h\gamma_k}{c_{k-1} - h\gamma_{k-1}}. \tag{4.41}$$

Relation (4.39) makes it evident that this map serves as a finite-difference approximation to the flow (2.2).

The matrices taking part in the Lax representation of this last map are

$$\mathcal{C}(c, d, \lambda) = \sum_{k=1}^{N} E_{kk} - h\lambda^{-1} \sum_{k=1}^{N} \gamma_k E_{k,k+1} \tag{4.42}$$

$$\mathcal{D}(c, d, \lambda) = \sum_{k=1}^{N} E_{kk} - h\lambda^{-1} \sum_{k=1}^{N} \delta_k E_{k,k+1}. \tag{4.43}$$

Theorem 6.

(i) $\mathcal{C}(c, d, \lambda) = \Pi_-\Big((I + hT_+^{-1}(c, d, \lambda))^{-1}\Big)$.

(ii) $\mathcal{D}(c, d, \lambda) = \Pi_-\Big((I + hT_-^{-1}(c, d, \lambda))^{-1}\Big)$.

(iii) The dynamical system (4.41) admits a Lax representation:

$$\mathcal{C}(t)L(t+h) = L(t)\mathcal{D}(t) \qquad \mathcal{C}(t)U(t+h) = U(t)\mathcal{D}(t) \tag{4.44}$$

which implies also

$$T_+(t+h) = \mathcal{C}^{-1}(t)T_+(t)\,\mathcal{C}(t) \qquad T_-(t+h) = \mathcal{D}^{-1}(t)T_-(t)\mathcal{D}(t). \tag{4.45}$$

Proof 4. The recurrence relations (4.37) are equivalent to the matrix factorization

$$L(c,d,\lambda) + hU(c,d,\lambda) = \mathcal{C}(c,d,\lambda)P_4(c,d,\lambda) \tag{4.46}$$

with the matrix

$$P_4(c,d,\lambda) = \sum_{k=1}^{N} \frac{c_{k-1}}{\gamma_{k-1}} E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1,k} \in \mathbf{G}_+.$$

This immediately implies statement (i), and, together with (4.32), implies statement (ii), because

$$I + hT_+^{-1} = \mathcal{C}P_4L^{-1} \qquad \text{and} \qquad P_4L^{-1} \in \mathbf{G}_+$$
$$I + hT_-^{-1} = L^{-1}\mathcal{C}P_4 = \mathcal{D}\widetilde{L}^{-1}P_4 \qquad \text{and} \qquad \widetilde{L}^{-1}P_4 \in \mathbf{G}_+.$$

In order to demonstrate (iii) we note that (4.44) is equivalent to

$$\gamma_{k-1}\widetilde{d}_k = d_{k-1}\delta_{k-1} \qquad \widetilde{d}_k - h\gamma_k = d_k - h\delta_{k-1} \tag{4.47}$$
$$\gamma_{k-1}\widetilde{c}_k = c_{k-1}\delta_k \qquad \widetilde{c}_k + h\gamma_k = c_k + h\delta_k \tag{4.48}$$

which is in turn equivalent to (4.41) and (4.38). The theorem is proved. □

Comparing now the results formulated in theorems 3–6 with theorem 2, we see that the maps (4.5), (4.17), (4.29) and (4.41) are symplectic with respect to the Poisson bracket (2.3), that the initial-value problem for the dynamical systems generated by these maps can be solved in terms of factorization of the matrices

$$(I + hT_\pm(0))^n \qquad \left(I - hT_\pm^{-1}(0)\right)^n \qquad \left((I - hT_\pm(0))^{-1}\right)^n \qquad \left((I + hT_\pm^{-1}(0))^{-1}\right)^n$$

respectively, and that the interpolating Hamiltonians for these discrete-time systems are given by

$$\mathrm{tr}(\Phi(T)) \qquad -\mathrm{tr}(\Phi(-T^{-1})) \qquad -\mathrm{tr}(\Phi(-T)) \qquad \mathrm{tr}(\Phi(T^{-1}))$$

respectively, where

$$\Phi(\xi) = h^{-1}\int_0^\xi \frac{\mathrm{d}\eta}{\eta}\log(1+h\eta) = \xi + \mathrm{O}(h).$$

5. Equations in physical variables

It is sometimes convenient to parametrize the variables c_k, d_k with the Poisson brackets (2.3) by means of canonically conjugated variables x_k, p_k:

$$d_k = \exp(p_k) \qquad c_k = g^2\exp(x_{k+1} - x_k + p_k)$$

($g^2 \in \mathbb{R}$ is a coupling constant). In terms of these variables

$$J_+ = \sum_{k=1}^{N} \exp(p_k)(1 + g^2\exp(x_{k+1} - x_k))$$
$$J_- = \sum_{k=1}^{N} \exp(-p_k)(1 + g^2\exp(x_k - x_{k-1})).$$

By means of $x_k, \dot{x}_k$ the variables d_k, c_k are parametrized in the case of J_+-flow as

$$d_k = \frac{\dot{x}_k}{1 + g^2 \exp(x_{k+1} - x_k)} \qquad c_k = g^2 \exp(x_{k+1} - x_k) d_k \tag{5.1}$$

and in the case of J_--flow as

$$d_k = -\frac{1 + g^2 \exp(x_k - x_{k-1})}{\dot{x}_k} \qquad c_k = g^2 \exp(x_{k+1} - x_k) d_k. \tag{5.2}$$

It is remarkable, although usually not stressed in the literature, that for *both* of the Hamiltonians $J_\pm$ the evolution of variables x_k is governed by the Newtonian equations of motion

$$\ddot{x}_k = \dot{x}_{k+1}\dot{x}_k \frac{g^2 \exp(x_{k+1} - x_k)}{1 + g^2 \exp(x_{k+1} - x_k)} - \dot{x}_k \dot{x}_{k-1} \frac{g^2 \exp(x_k - x_{k-1})}{1 + g^2 \exp(x_k - x_{k-1})} \qquad 1 \leqslant k \leqslant N. \tag{5.3}$$

We shall demonstrate that exactly the same phenomenon is found in the discrete-time case. More precisely, we shall demonstrate that after a suitable parametrization of the variables c_k, d_k *both* of the maps (4.5) and (4.17) are described by the 'Newtonian' equations of motion

$$\frac{\exp(x_k(t+h) - x_k(t)) - 1}{\exp(x_k(t) - x_k(t-h)) - 1} = \frac{\left(1 + g^2 \exp(x_{k+1}(t) - x_k(t))\right)}{\left(1 + g^2 \exp(x_{k+1}(t-h) - x_k(t))\right)} \times \frac{\left(1 + g^2 \exp(x_k(t) - x_{k-1}(t+h))\right)}{\left(1 + g^2 \exp(x_k(t) - x_{k-1}(t))\right)} \qquad 1 \leqslant k \leqslant N \tag{5.4}$$

and *both* of the maps (4.29) and (4.41) are described by the 'Newtonian' equations of motion

$$\frac{\exp(-x_k(t+h) + x_k(t)) - 1}{\exp(-x_k(t) + x_k(t-h)) - 1} = \frac{\left(1 + g^2 \exp(x_{k+1}(t+h) - x_k(t))\right)}{\left(1 + g^2 \exp(x_{k+1}(t) - x_k(t))\right)} \times \frac{\left(1 + g^2 \exp(x_k(t) - x_{k-1}(t))\right)}{\left(1 + g^2 \exp(x_k(t) - x_{k-1}(t-h))\right)} \qquad 1 \leqslant k \leqslant N. \tag{5.5}$$

Before we proceed to prove these assertions, some remarks are in order. First, as opposed to the continuous-time system (5.3), neither of the systems (5.4) and (5.5) is time reversible; instead, the reversal of time converts (5.4) into (5.5) and vice versa. Second, in the continuous-time limit they both tend to (5.3). Third, they both admit a simple non-relativistic limit: set $x_k(t) = q_k(t) + ct$ in (5.4) ($x_k(t) = q_k(t) - ct$ in (5.5)) with $c > 0$ playing the role of the speed of light; then in the limit $c \to \infty$ both of the equations tend to one and the same system:

$$\exp(q_k(t+h) - 2q_k(t) + q_k(t-h)) = \frac{1 + g^2 \exp(q_{k+1}(t) - q_k(t))}{1 + g^2 \exp(q_k(t) - q_{k-1}(t))} \qquad 1 \leqslant k \leqslant N$$

i.e. to the equations of motion of the discrete-time Toda lattice from [3]. Last, we note that (5.4) and (5.5) may be obtained by means of a certain limiting procedure starting from the discrete-time relativistic hyperbolic Calogero–Moser system [8], in just the same manner as is done for the continuous-time case in [9].

We give now explicitly the parametrizations of c_k, d_k leading to the Newtonian forms of equations of motion for all four of our maps separately. These formulae are to be compared with (5.1) and (5.2).

To do this for the map (4.5), we note that the equivalent equations (4.11) and (4.12) are identically satisfied by the following identifications:

$$\mathfrak{b}_k = \exp(\widetilde{x}_k - x_k)$$

$$\mathfrak{a}_k = \exp(\widetilde{x}_k - x_k)\frac{(1+g^2\exp(x_{k+1}-\widetilde{x}_k))}{(1+g^2\exp(x_{k+1}-x_k))}\,\frac{(1+g^2\exp(x_k-x_{k-1}))}{(1+g^2\exp(x_k-\widetilde{x}_{k-1}))}$$

$$d_k = \frac{(\exp(\widetilde{x}_k - x_k)-1)}{h(1+g^2\exp(x_{k+1}-x_k))}\,\frac{(1+g^2\exp(x_k-x_{k-1}))}{(1+g^2\exp(x_k-\widetilde{x}_{k-1}))} \tag{5.6}$$

$$\widetilde{d}_k = \frac{(\exp(\widetilde{x}_k - x_k)-1)}{h(1+g^2\exp(x_{k+1}-\widetilde{x}_k))} \tag{5.7}$$

and

$$c_k = g^2\exp(x_{k+1}-x_k)d_k \qquad \widetilde{c}_k = g^2\exp(\widetilde{x}_{k+1}-\widetilde{x}_k)\widetilde{d}_k. \tag{5.8}$$

The compatibility of expressions (5.6) and (5.7) immediately leads to (5.4).

Proceeding analogously with the map (4.17), we observe that the equivalent equations (4.23) and (4.24) are satisfied identically by the following identifications:

$$\mathfrak{d}_k = g^2\exp(x_{k+1}-\widetilde{x}_k)$$

$$\mathfrak{c}_k = g^2\exp(x_{k+1}-\widetilde{x}_k)\frac{(1-\exp(\widetilde{x}_{k+1}-x_{k+1}))}{(1-\exp(\widetilde{x}_k-x_k))}\,\frac{(1+g^2\exp(\widetilde{x}_k-\widetilde{x}_{k-1}))}{(1+g^2\exp(\widetilde{x}_{k+1}-\widetilde{x}_k))}$$

$$d_k = \frac{h(1+g^2\exp(x_k-\widetilde{x}_{k-1}))}{(1-\exp(\widetilde{x}_k-x_k))} \tag{5.9}$$

$$\widetilde{d}_k = \frac{h(1+g^2\exp(\widetilde{x}_k-\widetilde{x}_{k-1}))}{(1-\exp(\widetilde{x}_k-x_k))}\,\frac{(1+g^2\exp(x_{k+1}-\widetilde{x}_k))}{(1+g^2\exp(\widetilde{x}_{k+1}-\widetilde{x}_k))} \tag{5.10}$$

and (5.8). The compatibility of (5.9) and (5.10) leads again to (5.4).

For the map (4.29) the equivalent equations (4.35) and (4.36) are identically satisfied with the identifications

$$\beta_k = \exp(-\widetilde{x}_k + x_k)$$

$$\alpha_k = \exp(-\widetilde{x}_k + x_k)\frac{(1+g^2\exp(\widetilde{x}_{k+1}-x_k))}{(1+g^2\exp(\widetilde{x}_{k+1}-\widetilde{x}_k))}\,\frac{(1+g^2\exp(\widetilde{x}_k-\widetilde{x}_{k-1}))}{(1+g^2\exp(\widetilde{x}_k-x_{k-1}))}$$

$$d_k = \frac{(1-\exp(-\widetilde{x}_k + x_k))}{h(1+g^2\exp(\widetilde{x}_{k+1}-x_k))} \tag{5.11}$$

$$\widetilde{d}_k = \frac{(1-\exp(-\widetilde{x}_k + x_k))}{h(1+g^2\exp(\widetilde{x}_{k+1}-\widetilde{x}_k))}\,\frac{(1+g^2\exp(\widetilde{x}_k-\widetilde{x}_{k-1}))}{(1+g^2\exp(\widetilde{x}_k-x_{k-1}))} \tag{5.12}$$

and (5.8). This time the compatibility of (5.11) and (5.12) leads to the equation (5.5).

Finally, the equations (4.47) and (4.48), equivalent to the map (4.41), are satisfied identically under the identifications

$$\delta_k = g^2\exp(\widetilde{x}_{k+1}-x_k)$$

$$\gamma_k = g^2 \exp(\widetilde{x}_{k+1} - x_k) \frac{(\exp(-\widetilde{x}_{k+1} + x_{k+1}) - 1)}{(\exp(-\widetilde{x}_k + x_k) - 1)} \frac{(1 + g^2 \exp(x_k - x_{k-1}))}{(1 + g^2 \exp(x_{k+1} - x_k))}$$

$$d_k = \frac{h(1 + g^2 \exp(x_k - x_{k-1}))}{(\exp(-\widetilde{x}_k + x_k) - 1)} \frac{(1 + g^2 \exp(\widetilde{x}_{k+1} - x_k))}{(1 + g^2 \exp(x_{k+1} - x_k))} \tag{5.13}$$

$$\widetilde{d}_k = \frac{h(1 + g^2 \exp(\widetilde{x}_k - x_{k-1}))}{(\exp(-\widetilde{x}_k + x_k) - 1)} \tag{5.14}$$

and (5.8). Again, the compatibility of (5.13) and (5.14) leads to the system (5.5).

6. Conclusion

Our results suggest the following integrable discretization for an arbitrary flow of the relativistic Toda hierarchy with the Hamiltonian $\varphi(T)$: the desired map is given by the formulae of part (c) of theorem 2 with $f(T) = I + h\ \mathrm{d}\varphi(T)$ or $f(T) = (I - h\ \mathrm{d}\varphi(T))^{-1}$. (It could be difficult to express such a map explicitly in terms of c_k, d_k.)

In fact, this is a universal recipe for discretizing finite-dimensional integrable systems, whose phase space may be identified with an orbit of an r-matrix Poisson bracket on a Lie group. The author intends to describe other applications of this general approach in a separate paper.

The problem, however, lies in the fact that for some of the most beautiful known examples of integrable maps [1, 3, 5, 6] the phase space is an orbit of a *different* bracket to that for the underlying continuous-time system. For example, the discrete-time *non-relativistic* Toda lattice 'lives' on the same orbit as the continuous-time *relativistic* Toda lattice [3, 11]. Unfortunately, there seems to be no rule for identifying *a priori* the correct r-matrix bracket for beautiful discretizations, partly because of the non-rigorous nature of the notion 'beautiful'.

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